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ON A LINEAR INSTABILITY OF A PLANE PARALLEL COUETTE FLOW OF VISCOELASTIC FLUID

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Numerous investigations have shown that majority of real fluids cannot be described in terms of a constant viscosity (Newtonian) model. There exist various effects connected with elastic properties of these fluids, with the dependence of parameters on shear velocity etc.

It is these phenomena, inherently related to the nonnewtonian behavior of melts and solutions of polymers, that cause irregularities in their flow patterns and give rise to so called "elastic turbulence" [1 to 3].

All the flow irregularities exhibit a common characteristic feature, namely they appear at very small Reynolds numbers (they are very high viscosity fluids), when the usual hydrodynamic instability and turbulence cannot take place.

Assumption of the "elastic" character of this phenomenon is well supported by experimental data available, and several authors [4 to 6] use the critical value of a dimensionless parameter $\Gamma = \theta V L^{-1} = \eta G^{-1} V L^{-1}$, characterizing reversible elastic deformation of fluid, as the criterion of its appearance. Here η is the viscosity, G is the shear modulus, $\theta = \eta G^{-1}$ is the time of relaxation while V and L are characteristic velocity and linear dimension, respectively.

When the accumulated elastic deformation exceeds some critical value (of the order of 7), then the phenomenon described above takes place, and we can use this as a basis for another assumption. Just as the inertial forces in a viscous fluid, the elastic forces act, in viscoelastic fluids as an additional destabilizing factor (the connection between the elastic terms and additional nonlinearity in equations will be seen later on the model used). This in turn, leads to consideration of the possibility of a special "elastic" turbulence in viscoelastic fluids.

Following the example of [7] dealing with viscous fluids, we can consider the above problem together with that of hydrodynamic stability of the flow of viscoelastic fluid.

Several authors [8 to 12] have, in recent times, investigated the stability of simple flows of viscoelastic fluids, but only these stability changes were considered which were brought about by the action of elasticity when the fluid underwent small deviation from its Newtonian behavior, i. e. for small values of the elastic parameter Γ and only for these flows, which have already exhibited instability in case of a viscous fluid.

Unlike the previous papers on the stability of flows of viscoelastic fluids, this paper considers the linear instability of a plane-parallel Couette flow (which is, according to [7], linearly stable in a viscous fluid) at small Reynolds numbers and large values of parameter Γ (compare the phenomenon described above and investigated experimentally) i. e. under the conditions allowing the destabilizing influence of elasticity to manifest itself.

We shall in addition note, that in majority of papers investigating the influence of small amount of elasticity (low Γ) on the instability at large Reynolds numbers, destabilization had, in fact, occurred.

1. Equation of small plane perturbations. Let us consider a simple model of viscoelastic fluid, called Maxwell's model. It has two constants (viscosity and the time of relaxation) and describes the phenomenon of relaxation of stresses in a medium. Generalization of Maxwell's model to the case of high rates of deformation is found not to be single-valued [13] and we shall limit ourselves to a particular case (see Eq. (1. 2)) assuming, that this equation already contains these basic features which interest us.

Since the aim of this paper is to prove the instability of a plane-parallel Couette flow, therefore in the following we shall limit ourselves to the case of plane perturbations, disregarding the three-dimensional ones, although they are of undoubted interest when the critical value Γ_* (as $R \to 0$) is being determined, since for fluids with normal stresses Squire's theorem on the major role of plane perturbations does, apparently, not hold (see [14]).

So, we shall consider a plane shearing flow between two plane parallel plates, one of which is fixed, while the other moves with a given velocity V. We shall denote the distance between plates by L and in the following we shall employ the rectangular coordinate system with two axes lying in the fixed plane, x_2-axis perpendicular to it and x_1 coinciding with the direction of flow.

Equations of motion and continuity of incompressible fluid are

$$\frac{\partial v_i}{\partial t} + v_\alpha \frac{\partial v_i}{\partial x_\alpha} = - \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{i\alpha}}{\partial x_\alpha}, \qquad \frac{\partial v_\alpha}{\partial x_\alpha} = 0 \tag{1.1}$$

Here and in the following, kinematic pressure p, viscosity ν and stress tensor $\sigma_{i\alpha}$ are used and they were obtained by dividing the proper values by constant density.

We shall employ Maxwell's model with a single value of the stress relaxation time, extended to the case of large deformations by Oldroyd [13], as a model of viscoelastic fluid. Its rheological equations will be

$$\frac{\partial \sigma_{ij}}{\partial t} + v_\alpha \frac{\partial \sigma_{ij}}{\partial x_\alpha} - \frac{\partial v_i}{\partial x_\alpha} \sigma_{\alpha j} - \frac{\partial v_j}{\partial x_\alpha} \sigma_{\alpha i} + \frac{1}{\theta} \sigma_{ij} = \frac{\nu}{\theta} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{1.2}$$

and the combined solution of (1. 1) and (1. 2) in case of a steady plane shearing flow with usual no-slip conditions yields the following expressions for the velocity vector and the stresses

$$\mathbf{v} = (VL^{-1}x_2, 0), \qquad \|\sigma_{ij}\| = \nu \begin{pmatrix} 2\theta V^2L^{-2} & VL^{-1} \\ VL^{-1} & 0 \end{pmatrix} \tag{1.3}$$

Let us superimpose on this steady solution, small plane perturbations, which are exponential functions of time and the coordinate x_1.

Neglecting the terms nonlinear in perturbations, we can easily reduce the obtained system to a single fourth order equation.

Selecting the length L and velocity V as characteristic magnitudes (dimension of mass was eliminated by dividing everything by density), we can obtain that equation in dimensionless form

$$[y_*^2D^2 - 2y_*D + 2 - \alpha^2y_*^2]\,[D^2 + 2i\alpha\,\Gamma D - \alpha^2 - 2\alpha^2\Gamma^2]\,v + $$
$$+ \alpha^2\Gamma Ry_*^3\,(y - c)\,[D^2 - \alpha^2]v = 0 \tag{1.4}$$

where $v(y)\exp[i\alpha\,(x - ct)]$ is the ($y = L^{-1}x_2$)-component (proportional to the stream function) of perturbation velocity,

$$x = L^{-1}x_1, \quad D = d\,/\,dy, \; y_* = y - c - i\alpha^{-1}\Gamma^{-1}$$

and finally $R = VL\,\nu^{-1}$ and $\Gamma = \theta VL^{-1}$ are the Reynolds number and a dimensionless "elasticity parameter", the latter indicating the amount of accumulated elastic deformation in the shearing flow.

It should be noted that the linearized stability problem can be solved as an initial value problem, using Laplace transforms with respect to time and Fourier transforms along the longitudinal coordinate.

Moreover it can be shown, that such a formulation of a problem is, as in the case of viscous fluid [7], equivalent to the analysis of instability in terms of elementary wave solutions. New solutions of an eigenvalue problem emerging here corresponding to the continuous spectrum, decay with time.

We shall mention an interesting circumstance influencing our choice of model equations of state of viscoelastic fluid. Out of all possible generalizations of Maxwell's model to the case of high rates of deformation, we have chosen the equations of a "contravariant" model (1.2).

Equations of a "covariant" model

$$\frac{\partial\sigma_{ij}}{\partial t} + v_\alpha\frac{\partial\sigma_{ij}}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_i}\sigma_{\alpha j} + \frac{\partial v_\alpha}{\partial x_j}\sigma_{\alpha i} + \frac{1}{\theta}\sigma_{ij} = \frac{\nu}{\theta}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

present another possible generalization.

For a steady Couette flow, the stress tensor differs from (1.3) and has the form

$$\|\sigma_{ij}\| = \nu\begin{pmatrix} 0 & VL^{-1} \\ VL^{-1} & -2\theta V^2L^{-2} \end{pmatrix}$$

nevertheless equations of plane perturbations are exactly the same as in case of a contravariant model, i. e. Eqs. (1.4).

The latter should be solved under the usual boundary no-slip conditions

$$v\,(0) = Dv\,(0) = v\,(1) = Dv\,(1) = 0 \tag{1.5}$$

Since (1.4) with (1.5) cannot be solved in its explicit form, we shall consider several asymptotic solutions of this equation. In the following we shall only consider the case $R \ll 1$ which is of practical interest, and which corresponds to the case of flow of elastic fluids of high viscosity. We shall also assume that the elasticity parameter Γ is not small ($\Gamma \gtrsim 1$). As we know, in such problems we must determine the connection between the parameter c, taken as an eigenvalue of the problem (1, 4) and (1, 5) and other parameters of the problem which are: wave number α, Reynolds number R and the elastic parameter Γ. We shall therefore consider several cases, making various assumptions about the quantity $|c|$.

2. Linear stability of a plane, inertialess Couette flow of Maxwellian fluid. Let $|c| \lesssim 1$. Taking the previous assumption that $R \ll 1$ into account, we easily see that in this case we can neglect, in (1, 4), terms containing Reynolds number R. Eq. (1, 4) will then become

$$(y_*^2 D^2 - 2y_* D + 2 - \alpha^2 y_*^2)(D^2 + 2i\alpha\Gamma D - \alpha^2 - 2\alpha^2\Gamma^2)v = 0$$

Basic system of solutions of this equation is

$$v_1 = (y - c)e^{\alpha y}, \qquad v_2 = (y - c)e^{-\alpha y}$$

$$v_3 = \exp\left[\alpha\Gamma\left(-i + \sqrt{1 + \Gamma^{-2}}\right)y\right], \qquad v_4 = \exp\left[-\alpha\Gamma\left(i + \sqrt{1 + \Gamma^{-2}}\right)y\right]$$

Fulfilling the conditions (1, 5) is equivalent to equating to zero the characteristic determinant ($\beta = (1 + \Gamma^{-2})^{1/2}$)

$$\begin{vmatrix} -c & -c & 1 & 1 \\ 1 - \alpha c & 1 + \alpha c & \alpha\Gamma(\beta - i) & -\alpha\Gamma(\beta + i) \\ (1 - c)e^{\alpha} & (1 - c)e^{-\alpha} & e^{\alpha\Gamma(\beta - i)} & e^{-\alpha\Gamma(\beta + i)} \\ [1 + \alpha(1 - c)]e^{\alpha} & [1 - \alpha(1 - c)]e^{-\alpha} & \alpha\Gamma(\beta - i)e^{\alpha\Gamma(\beta - i)} & -\alpha\Gamma(\beta + i)e^{-\alpha\Gamma(\beta + i)} \end{vmatrix} = 0$$

which on expansion, yields an equation quadratic in c, whose solution is

$$c = {}^1/_2 + iA \pm \sqrt{{}^1/_4 - A^2 + B}$$

$$A = \frac{\alpha\Gamma \sinh\alpha \sinh\alpha\beta\Gamma - \alpha^2\beta\Gamma \sin\alpha\Gamma}{2\alpha^2\beta\Gamma\left[\cosh\alpha \cosh\alpha\beta\Gamma - \cos\alpha\Gamma - \beta\Gamma \sinh\alpha \sinh\alpha\beta\Gamma\right]} \tag{2.1}$$

$$B = \frac{\alpha\beta\Gamma \sinh\alpha \cosh\alpha\beta\Gamma + \alpha^2\beta\Gamma\cos\alpha\Gamma - (\alpha\cosh\alpha + \sinh\alpha)\sinh\alpha\beta\Gamma}{2\alpha^2\beta\Gamma\left[\cosh\alpha \cosh\alpha\beta\Gamma - \cos\alpha\Gamma - \beta\Gamma \sinh\alpha \sinh\alpha\beta\Gamma\right]}$$

It is easy to confirm that the numerator of A in (2, 1) is positive and becomes equal to zero only when $\alpha = 0$ or $\Gamma = 0$; denominator of A is a negative and becomes equal to zero also when $\alpha = 0$ or when $\Gamma = 0$, hence $A < 0$.

Then, it follows from (2, 1) that the condition of instability $\operatorname{Im} c \geq 0$ is realized only when $B \geq -\frac{1}{4}$. The latter will be true, if the function

$$\Phi(\alpha, z) = 2z \sinh\alpha \cosh z + \alpha z\left(\cosh\alpha \cosh z + \cos\sqrt{z^2 - \alpha^2}\right) -$$
$$- z^2 \sinh\alpha \sinh z - 2\sinh z\,(\sinh\alpha + \alpha\cosh\alpha), \qquad z = \alpha(1 + \Gamma^2)^{1/2} \tag{2.2}$$

becomes equal to zero.

Definition of z infers that Φ is defined in the region $\{\alpha \geq 0,\ z \geq \alpha\}$ (here and in the following, as we can easily show, it is sufficient to consider the values $\alpha \geq 0$ only). Differentiation of $\Phi(\alpha, z)$, with respect to z, yields

$$\Phi_z' = \alpha\left(\cos\sqrt{z^2-\alpha^2} - \frac{z^2\sin\sqrt{z^2-\alpha^2}}{\sqrt{z^2-\alpha^2}} - \cosh\alpha\cosh z\right) +$$

$$+\alpha z^2\cosh\alpha\cosh z\left(\frac{\tanh z}{z} - \frac{\tanh\alpha}{\alpha}\right) \tag{2.3}$$

Since the function $z^{-1}\tanh z$ is monotonously decreasing when $z > 0$, then the assumption that $z > \alpha$ implies that the second term in (2.3) is negative. Taking into account an additional fact that the first term of (2.3) is negative, we conclude that $\Phi_z' < 0$ for fixed $\alpha > 0$. Since it follows from (2.2) that when $z = \alpha$

$$\Phi\,|_{z=\alpha} = 2\,(\alpha^2 - \sinh^2\alpha) < 0$$

then $\Phi(\alpha, z) < 0$ in the region $\{\alpha > 0,\ z \geq \alpha\}$ and this imples that Im$c < 0$ when $\alpha > 0$, i.e. solution of the problem is stable in the inertialess approximation.

3. Perturbations in presence of high frequency oscillations. We shall now consider the case when $|c| \gg 1$. Then, as we can easily see from (1.4), both terms in this equation will be of the same order, if

$$\varepsilon\alpha\Gamma c \sim 1, \qquad \varepsilon = R^{1/2}\Gamma^{-1/2} \ll 1 \tag{3.1}$$

Now, $|c| \gg 1$ implies that $\alpha\Gamma \ll \varepsilon^{-1}$. We shall seek the eigenvalue c and solution v in the form of a series used in the perturbation theory

$$c = \varepsilon^{-1}s, \qquad s = s_0 + \varepsilon s_1 + \ldots,\quad v = v_0 + \varepsilon v_1 + \ldots \tag{3.2}$$

Expansion of the coefficients in (1.4) into series in ε, yields

$$L_0 v_0 \equiv (D^2 - \alpha^2)\ \ [D^2 + 2i\alpha\Gamma D + \alpha^2\Gamma^2\,(s_0^2 - 2 - \Gamma^{-2})]\ \ v_0 = 0 \tag{3.3}$$

$$L_0 v_1 = [2(i/s)\,D\,(D^2 + 2i\alpha\Gamma D - \alpha^2 - 2\alpha^2\Gamma^2) + \alpha\Gamma s\,(1 + 2\alpha\Gamma y)\,(D^2 - \alpha^2)]v_0$$

which can be solved with initial conditions (1.5). In the expression for v_1 we have for convenience retained the parameter $s = s_0 + \varepsilon s_1 + \ldots$ with the understanding that this expansion will be included in the characteristic determinant during the final stage of computations.

Let us inspect the solution of our problem in the zero approximation. Basic set of solutions has, in this case, the form

$$e^{\alpha y},\quad e^{-\alpha y},\quad e^{i\alpha\Gamma(\gamma-1)y},\quad e^{-i\alpha\Gamma(\gamma+1)y},\quad \gamma = (s_0^2 - 1 - \Gamma^{-2})^{1/2}) \tag{3.4}$$

Equating to zero the characteristic determinant of the system, we obtain

$$\Delta_0 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & -\alpha & i\alpha\Gamma(\gamma-1) & -i\alpha\Gamma(\gamma+1) \\ e^{\alpha} & e^{-\alpha} & e^{i\alpha\Gamma(\gamma-1)} & e^{-i\alpha\Gamma(\gamma+1)} \\ \alpha e^{\alpha} & -\alpha e^{-\alpha} & i\alpha\Gamma(\gamma-1)\,e^{i\alpha\Gamma(\gamma-1)} & -i\alpha\Gamma(\gamma+1)\,e^{-i\alpha\Gamma(\gamma+1)} \end{vmatrix} = 0 \tag{3.5}$$

which, on expansion, yields the dispersion relation

$$[1 + \Gamma^2\,(1 - \gamma^2)]\,\sin\alpha\Gamma\gamma\,\sinh\alpha - 2\Gamma\gamma\,(\cos\alpha\Gamma\gamma\,\cosh\alpha - \cos\alpha\Gamma) = 0 \tag{3.6}$$

possessing no solutions on the imaginary axis γ. All its solutions except for five trivial ones

$$\gamma = 0, \qquad \gamma = \pm 1 \pm i\Gamma^{-1} \tag{3.7}$$

expressing the linear dependence between the solutions (3.4), lie on the real axis, spaced at the intervals approximately equal to π (except four of such intervals). Applying Rouche's theorem on roots of analytic functions to (3.6) we see, that the roots listed above account for all the roots γ_k of (3.6). Thus, apart from the trivial ones, all the roots γ_k are real, the corresponding eigenvalues s_{0k} are purely real, and to obtain the solution of our problem we must investigate the following approximation of the perturbation theory.

As we mentioned before, linearly dependent solutions of the first equation of (3.3) correspond to the roots of (3.7). It can be shown, by constructing in the usual manner linearly independent equations corresponding to these roots, that the characteristic determinant cannot, in this case, become equal to zero, i.e. numbers (3.7) will not be the eigenvalues of the problem.

Consider now the solution of our problem in the first approximation. Eqs. (3.3) easily yield the basic set of solutions, which is

$$v_1 = e^{\alpha y}(1 + \varepsilon a y), \qquad v_2 = e^{-\alpha y}(1 + \varepsilon \bar{a} y)$$

$$v_3 = e^{i\alpha\Gamma(\gamma-1)y}[1 + \varepsilon(b^+ y + e y^2)]$$

$$v_4 = e^{-i\alpha\Gamma(\gamma+1)y}[1 + \varepsilon(b^- y + \bar{e} y^2)]$$

$$a = \frac{2}{s}\,\frac{\Gamma^{-1} - i}{s^2 + 2i\Gamma^{-1} - 2}, \qquad e = -i\frac{\alpha\Gamma s}{2\gamma}$$

$$b^+ = (1-\gamma)\frac{s}{2\gamma^2}\,\frac{s^2 - 4\gamma}{s^2 - 2\gamma}, \qquad b^-(s,\gamma) = b^+(s,-\gamma) \tag{3.8}$$

where $\bar{a}$ and $\bar{e}$ are the complex conjugate of a and e.

If the boundary conditions (1.5) are fulfilled, then the characteristic determinant becomes equal to zero

$$\Delta = \begin{vmatrix} v_{10} & v_{20} & v_{30} & v_{40} \\ v_{10}' & v_{20}' & v_{30}' & v_{40}' \\ v_{11} & v_{21} & v_{31} & v_{41} \\ v_{11}' & v_{21}' & v_{31}' & v_{41}' \end{vmatrix} = 0 \tag{3.9}$$

Usual system of indices is adopted here, first subscript in v_{ik} corresponding to the consecutive number of solution, the second one to the value of $y = 0,1$. A prime denotes a derivative with respect to y.

The following computation scheme is adopted. Inserting (3.8) into (3.9) and expanding the determinant into a series in ε, we obtain

$$\Delta(\alpha, \Gamma, s, \varepsilon) = \Delta_0(\alpha, \Gamma, s) + \varepsilon\Delta_1(\alpha, \Gamma, s) + \ldots \tag{3.10}$$

Let us recall that, by (3.2), $s = s_0 + \varepsilon s_1 + \ldots$. Then, expansion of s in (3.10) yields

$$\Delta(\alpha, \Gamma, s, \varepsilon) = \Delta_0(\alpha, \Gamma, s_0) + \varepsilon\{s_1 \partial/\partial s\,\Delta_0(\alpha, \Gamma, s)|_{s_0} + \Delta_1(\alpha, \Gamma, s_0)\} + O(\varepsilon^2) = 0$$

Putting $\Delta_0(\alpha, \Gamma, s_0) = 0$, we obtain

$$s_1 = -\Delta_1(\alpha, \Gamma, s_0)\,[\partial/\partial s\,\Delta_0(\alpha, \Gamma, s_0)]^{-1} \tag{3.11}$$

Expanding (3.10) into a series in ε we easily see that the form of $\Delta_1(\alpha, \Gamma, s_0)$, suitable for computation, is

$$\Delta_1 = \left[a\frac{\partial\Delta_0}{\partial m_1}\bar{a} + \frac{\partial\Delta_0}{\partial m_2} + b^+\frac{\partial\Delta_0}{\partial m_3} + b^-\frac{\partial\Delta_0}{\partial m_4} + e\frac{\partial^2\Delta_0}{\partial m_3^2} + \bar{e}\frac{\partial^2\Delta_0}{\partial m_4^2}\right]_{m=m_k}.$$

$$\Delta_0(m_k) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ m_1 & m_2 & m_3 & m_4 \\ e^{m_1} & e^{m_2} & e^{m_3} & e^{m_4} \\ m_1 e^{m_1} & m_2 e^{m_2} & m_3 e^{m_3} & m_4 e^{m_4} \end{vmatrix} =$$

$$= (m_4 - m_3)(m_2 - m_1)(e^{m_1+m_2} + e^{m_4+m_3}) + (m_3 - m_1)(m_4 - m_2)(e^{m_1+m_3} + e^{m_2+m_4}) + (m_3 - m_2)(m_4 - m_1)(e^{m_1+m_4} + e^{m_3+m_2})$$

$$m_1^\circ = \alpha, \quad m_2^\circ = -\alpha, \quad m_3^\circ = i\alpha\Gamma(\gamma - 1), \quad m_4^\circ = -i\alpha\Gamma(\gamma + 1)$$

After a lengthy calculation following the scheme based on (3.11), we obtain the required connection between $\operatorname{Im} s_1$ and the parameters α and Γ of the problem, as a pair of transcendental Eqs.

$$2\alpha\Gamma \operatorname{Im} s_1 + 1 = \frac{4\Gamma^4\gamma^2}{1 + \Gamma^2 + \Gamma^2\gamma^2} \frac{\alpha\gamma \sin \alpha\Gamma (\sinh\alpha \sin \alpha\Gamma\gamma)^{-1} - 1}{(\alpha\Gamma\gamma \cot \alpha\Gamma\gamma - 1)(1 + \Gamma^2 - \Gamma^2\gamma^2) + 2\Gamma^2\gamma^2(\alpha \coth \alpha - 1)} \tag{3.12}$$

$$F = (1 + \Gamma^2 - \Gamma^2\gamma^2) \sinh \alpha \sin \alpha\Gamma\gamma - 2\Gamma\gamma(\cosh\alpha \cos \alpha\Gamma\gamma - \cos \alpha\Gamma) = 0$$

Let us examine the basic properties of these Eqs. Dividing the second Eq. of (3.12) by $\alpha^{-2}\sinh\alpha \sin \alpha\Gamma\gamma$ ($\sin \alpha\Gamma\gamma \neq 0$, apart from a trivial case of $\gamma = 0$) and putting $\alpha\Gamma\gamma = z$, we obtain

$$\alpha^2(1 + \Gamma^2) - z^2 = 2\alpha z \coth \alpha \left(\cot z - \frac{\cos \alpha\Gamma}{\cosh\alpha \sin z} \right) \tag{3.13}$$

Since for $z = 0$ we have

$$\alpha^2(1 + \Gamma^2) > 2\alpha \coth \alpha \left(1 - \frac{\cos \alpha\Gamma}{\cosh\alpha} \right)$$

we easily see that the first root of (3.13) $z_1 < \pi$. When $z^2 \ll \alpha^2(1 + \Gamma^2)$ $(z \gg \pi)$ the roots $z_k(\alpha, \Gamma)$ of Eq. (3.13) are spaced at the intervals approximately equal to π.

Analogous situation arises when $z^2 \gg \alpha^2(1 + \Gamma^2)$, and $z_k \to k\pi$ as $k \to \infty$. According to the Rouche's theorem mentioned before, if $k\pi < z < (k + 1)\pi$ and $z^2 \gg \alpha^2(1 + \Gamma^2)$, then there are $(k - 2)$ roots of this Eq. on the interval $[0, z]$. Two roots are lost in the vicinity of a point $z = \alpha(1 + \Gamma^2)^{1/2}$ (they are included amongst the trivial roots (3.7)). Applying again the Rouche's theorem we can easily show that

$$\left[\frac{\partial}{\partial z} \frac{F(z, \alpha, \Gamma)}{\sin z} \right]_{z=\alpha\Gamma\gamma} = \left[\frac{F_z'}{\sin z} \right]_{z=\alpha\Gamma\gamma} > 0$$

in which F is given by the second expression of (3.12).

We can show however by direct calculation that the denominator of the right-hand side of the first Eq. of (3.12) differs from $F'_z(\alpha, \Gamma, \gamma)(\sin \alpha\Gamma\gamma)^{-1}$ by a positive multiplier only, therefore it is also positive. From this it follows directly that $\operatorname{Im} s_1 < 0$ on the lines $\alpha\Gamma = k\pi$, i.e. on these lines we have stability. By means of more cumbersome calculations it might be shown that we also have stability on the lines $\alpha\Gamma = k\pi/2$. This supports an assumption, not fully proved, that the problem is stable in the considered approximation.

Thus we have shown that the solution of our problem is stable on the plane $R^{\frac{1}{2}}\Gamma^{-\frac{1}{2}} = 0$ of the parametric space $R^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}$, α, Γ. This apparently occurs also on the plane $R^{1/2}\Gamma^{-1/2} = \varepsilon \ll 1$ of the region $\alpha\Gamma \ll \varepsilon^{-1}$. In the next Section we shall consider such an asymptotic expansion, which will allow us to investigate the region of large values of $\alpha\Gamma$ $(\alpha\Gamma \sim \varepsilon^{-1})$.

4. Proof of the instability. Assuming, as before, that the parameter $\epsilon = R^2\Gamma^{-2}$ is small, we shall consider the region of variation of the remaining parameters α, Γ and c, the region connected with the size of ϵ in the following manner

$$\alpha\Gamma \sim \Gamma^2 \sim \varepsilon^{-1} \gg 1$$

Since the terms of (1, 4) containing R disappear when $|c| \lesssim 1$ and the equation reduces to the stable case already considered, we assume that $|c| \gg 1$ and $c = \epsilon^{-1}s$, where

$$s = s_0 + \varepsilon s_1 + \ldots$$

Then, expanding the coefficients of (1, 4) and neglecting the infinitesimals of the order higher than second, we obtain

$$D^4v + 2i\alpha\Gamma D^3v + \alpha^2(-2-2\Gamma^2 + \Gamma^2s^2 - 2\varepsilon\Gamma^2ys)D^2v -$$
$$-2\alpha^2\Gamma\left(i\alpha + \varepsilon\frac{2\Gamma}{s}\right)Dv - \alpha^4(\Gamma^2s^2 - 2\Gamma^2 - 1 - 2\varepsilon\Gamma^2ys)\,v = 0 \qquad (4.1)$$

If only the biggest terms were retained, then the equation would become the first Eq. of (3, 3), already considered.

By the previous argument, all the roots $\gamma_k = \sqrt{s_{0k}{}^2 - 1 - \Gamma^{-2}}$ lie on the real axis and, apart from two roots, have an approximate separation of $\pi\,(\alpha\Gamma)^{-1}$.

Therefore, the imaginary part of s should be sought in the next approximation and we can investigate the following particular case (with a suitable choice of parameter $\alpha\Gamma$, when γ coincides with one of the roots γ_k)

$$s^2 = 2\,(1 + \varepsilon\,\xi + \ldots)$$

Then, with the previous accuracy, Eq. (4, 1) becomes

$$D^4v + 2\,i\alpha\Gamma D^3v - 2\alpha^2\,(1 \pm \varepsilon\Gamma^2y\sqrt{2} - \varepsilon\Gamma^2\xi)\,D^2v - 2\alpha^2\Gamma\,(i\alpha \pm \varepsilon\Gamma\sqrt{2})\,Dv +$$
$$+\alpha^4\,(1 \pm 2\sqrt{2}\varepsilon\Gamma^2y - 2\varepsilon\Gamma^2\xi)\,v = 0 \qquad (4.2)$$

Since the latter has nonanalytic solutions (in terms of the parameter $\epsilon \ll 1$), we shall seek them in the form

$$v = \exp\int g\,(y)\,dy$$

Further, inserting for g the expansion

$$g = \alpha^2g_{-2} + \alpha g_{-1} + g_0 + \alpha^{-1}g_1 + \alpha^{-2}g_2 + \ldots$$

we shall obtain, within the previous accuracy (up to the order of ϵ), the following solutions:

$$v_1 = \exp\left[-2i\alpha\Gamma y + i\frac{\alpha}{2\Gamma}(1 \pm \varepsilon\Gamma^2\sqrt{2}\,y - 2\varepsilon\Gamma^2\xi)\,y\right]\left(1 + O\left(\frac{1}{\alpha}\right)\right)$$
$$v_2 = \exp\,[\alpha y]\,(1 + O\,(1/\alpha))$$
$$v_3 = \exp\,[-\alpha y]\,(1 + O\,(1/\alpha)) \qquad (4.3)$$
$$v_4 = \exp\left[-i\frac{\alpha}{2\Gamma}(1 \pm \varepsilon\Gamma^2\sqrt{2}\,y - 2\varepsilon\Gamma^2\xi)\,y\right]\left(1 + \frac{1}{\alpha} + O\left(\frac{1}{\alpha^2}\right)\right)$$

Constructing from these solutions a characteristic determinant (3, 9) and equating it to zero, we obtain

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & -\alpha & \eta+\lambda & -\eta+\mu \\ e^{\alpha} & e^{-\alpha} & e^{\eta-i\alpha\Gamma} & e^{-\eta-i\alpha\Gamma} \\ \alpha e^{\alpha} & -\alpha e^{-\alpha} & (\eta+\mu)e^{\eta-i\alpha\Gamma} & (-\eta+\lambda)e^{-\eta-i\alpha\Gamma} \end{vmatrix} = 0$$

where

$$\lambda = -i\alpha\Gamma \pm {}^1/_2 i\alpha\Gamma\varepsilon\sqrt{2}$$

$$\mu = -i\alpha\Gamma \mp {}^1/_2 i\alpha\Gamma\varepsilon\sqrt{2}, \qquad \eta = i\alpha\Gamma - {}^1/_2 i\alpha\Gamma^{-1}(1 \pm \sqrt{2}\varepsilon\Gamma^2 - 2\varepsilon\Gamma^2\xi)$$

If we now neglect, in this determinant, the terms of the order of unity and of α in comparison with e^{α} in the first two columns and terms of the order of unity in comparison with $\alpha\Gamma$, then the simplified determinant will become

$$\begin{vmatrix} 0 & 1 & 1 & 0 \\ 0 & -\alpha & \eta+\lambda & 1 \\ 1 & 0 & e^{\eta-i\alpha\Gamma} & 0 \\ \alpha & 0 & (\eta+\mu)e^{\eta-i\alpha\Gamma} & e^{-\eta-i\alpha\Gamma} \end{vmatrix} = 0$$

which on expansion yields

$$(\eta - i\alpha\Gamma)\sinh\eta = (\alpha \pm i\alpha\Gamma\varepsilon / \sqrt{2})\cosh\eta \tag{4.4}$$

Putting $\xi = \xi_r + i\xi_i$ we obtain $\eta = \eta_r + i\eta_i$ where

$$\eta_r = -\alpha\Gamma\varepsilon\xi_i, \qquad \eta_i = \alpha\Gamma - \frac{\alpha}{2\Gamma}(1 \pm \sqrt{2}\varepsilon\Gamma^2 - 2\varepsilon\Gamma^2\xi_r) \tag{4.5}$$

Further, assuming that

$$c = \pm\varepsilon^{-1}\sqrt{2}\sqrt{1 + {}^1/_2\varepsilon\xi} \approx \pm\varepsilon^{-1}\sqrt{2} \pm {}^1/_4\sqrt{2}\,\xi + O(\varepsilon)$$

we shall have

$$\operatorname{Im} c = \pm\frac{\sqrt{2}}{4}\xi_i = \mp\frac{\eta_r}{\alpha\Gamma\varepsilon}\frac{\sqrt{2}}{4} \tag{4.6}$$

Separating (4.4) into real and imaginary parts, we obtain a system of transcendental equations, easily reducible to

$$\tan\eta_i = \frac{\pm\alpha\Gamma\varepsilon/\sqrt{2} - (\eta_i - \alpha\Gamma)\tanh\eta_r}{\eta_r - \alpha\tanh\eta_r}$$
$$\tanh 2\eta_r = 2\frac{\alpha\eta_r \pm \alpha\Gamma\varepsilon(\eta_i - \alpha\Gamma)/\sqrt{2}}{\alpha^2 + \eta_r^2 + {}^1/_2\alpha^2\Gamma^2\varepsilon^2 + (\eta_i - \alpha\Gamma)^2} \tag{4.7}$$

From the second Eq. of (4.7) we can infer, in accord with the previous assumption that $(\eta_i - \alpha\Gamma) \sim 1$ and $\varepsilon\alpha\Gamma \sim 1$, this equation can be written in a simple form

$$\tanh 2\eta_r \approx \frac{2\alpha\eta_r}{\alpha^2 + \eta_r^2}, \qquad \text{or} \qquad \alpha \approx \eta_r\frac{\cosh 2\eta_r \pm 1}{\sinh 2\eta_r}$$

By the previous assumption $\alpha \gg 1$, therefore $\eta_r \approx \alpha$. First Eq. of (4.7) yields the following expression for η_i

$$\tan\eta_i \approx (2\alpha)^{-1}e^{2\alpha}\,[\pm\alpha\Gamma\varepsilon/\sqrt{2} - (\eta_i - \alpha\Gamma)\tanh\alpha]$$

and we easily see that its roots are $(\eta_i - \alpha\Gamma) \sim 1$. From (4.6) we have

$$\operatorname{Im} c \approx \pm\frac{1}{2\sqrt{2}\Gamma\varepsilon} \tag{4.8}$$

and from it we see that in the parametric space under consideration, increasing perturbations exist, with an amplitude proportional to

$$\exp \frac{\alpha t}{2\sqrt{2R\Gamma}}$$

Thus we have shown that in the parameteric (α, Γ, $R^{1/2}\Gamma^{-1/2}$) space (on the plane $R^{1/2}\Gamma^{-1/2} = \varepsilon \ll 1$ in the region $\alpha\Gamma \sim \varepsilon^{-1}$, $\alpha \sim \Gamma$) instability occurs, with sufficiently large index of growth of perturbations. Question of boundaries of the region of instability and of the critical values of parameters Γ and R characterizing the onset of instability, remains open.

Our results show that unlike the case of viscous fluid for which plane-parallel Couette flow apparently exhibits linear stability, in case of our model of viscoelastic fluid such a flow is found unstable. This instability caused by the elasticity of fluid differs basically from the usual instability of a viscous fluid by the fact, that it occurs at quite small Reynolds numbers and high wave numbers.

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